Nonlinear geometric models

Theses of Ph.D. Dissertation

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Budapest, 2015
1 Motivation

Representation of shapes is of primary importance in every geometric problem. The answer to a geometric query may be trivial in one representation, and might require a more elaborate mathematical process in the other.

In computer-aided design (CAD), and in many differential geometric topics, parametric representation of shapes has become the most widespread formulation, which defines the shape as the image of a mapping.

In the CAD domain, the designer is mostly concerned with the geometry of the shape, that is, the image of the parametric function. The actual mapping between the parametric domain and the shape may not necessarily be specified, and as long as the image stays the same, or modified within bounds, this mapping may be changed.

Still, in many applications the mapping itself is a crucial component as well. For instance, tool paths can be represented by parametric curves in NC machining. Here, the parametrization determines how fast the tool travels along the path. This motion, however, is subject to physical constraints, such as that the tool cannot move at arbitrary speeds, and the parametrization has to conform to these restrictions.

The most widespread parametric representations in use, such as Bézier curves, B-splines, NURBS, etc. do not provide means to trivially separate the geometric properties of the shape from parametrization.

One of the goals of this thesis is to present a basis-independent formulation to carry out this separation.

Once geometric constraints are separated from the degrees of freedom of parametrization, the thesis discusses the reconstruction of prescribed geometric quantities at knots, that is, the problem of geometric Hermite (GH) interpolation.

Classical Hermite interpolation creates curves that reconstruct prescribed positional and derivative vector data at parametric endpoints. Geometric Hermite (GH) interpolation aims at finding curves such that they reconstruct prescribed geometric invariants, that is, parametrization independent data.

The first published industrial application of a second order GH interpolation scheme – that is, reconstruction of position, tangent direction, normal vector, and curvature values at endpoints – is due to Klass, in 1983 [16]. He
created a cubic integral polynomial curve to approximate the offset of a given curve by reconstructing the aforementioned quantities, sampled from the exact offset. Klass found, via experiments, that the second order GH interpolant provided a high accuracy approximation to the offset.

The exact accuracy of this approximation was not verified algebraically, however, until 1987. That year – independently of Klass –, de Boor, Höllig, and Sabin investigated the same problem of reconstructing second order geometric invariants at knots with cubic splines [15]. They found that such an approximation is sixth order accurate. They also addressed the issue of existence, and proved that if the progenitor curve – from which the second order GH data were sampled – is smooth enough, its curvature does not vanish, and sample points are close, there is always a cubic solution to this problem. This was also the paper that introduced the term geometric Hermite interpolation to refer to these types of problems.

Both of the above papers handled GH interpolation as a tool of curve approximation, thus the existence conditions were formulated in accordance with that framework. However, these are not tangible concepts from a design point of view: there is no underlying curve in that setting, instead, second order GH data are specified by the user, either directly or indirectly.

Schaback was the first to give a purely geometric set of existence conditions for second order GH interpolation in [18]. He presented necessary and sufficient conditions for the existence of quartic and cubic interpolants, and shown when only a quintic polynomial can achieve reconstruction. This has also emphasized that the interpolation problem investigated by Klass and de Boor et. al. are analogous to the quintic Hermite interpolation: the reconstruction of positional, first, and second derivative data.

Indeed, it was Mørken who pointed out that this cubic second order GH interpolant can be considered as a reparametrization of a quintic polynomial such that the coefficients of the quintic and quartic terms become zero. He provided a mathematical analysis of when such a degree reducing parametrization is possible in [17]. He did preliminary work on generalizing his parametric approximation framework to surfaces as well.

The main goal of this thesis is to provide a general formalization of geometric Hermite interpolation of curves and surfaces in a general class of bases. The formulation presented here allows us to unify the design and approximation-
Figure 1: Design and refinement process using control circles to define cubic Bézier curves. From left to right: control circle hierarchy, Bézier segments connected, application of curve-parameter dependent brush.

It is important to emphasize that the geometric model presented here is not a substitute for the existing representations, already implemented and dominating in CAD systems. Instead, it acts as an interface to separate the geometric and the parametrization components of the control data of an arbitrary underlying representation, whenever possible. As an example, Figure 1 illustrates how purely geometric input can be used to provide point, tangent, curvature input for a design process that yields cubic Bézier curves [3], [14].

2 Theses

The first thesis summarizes my results on the formulation of general geometric Hermite interpolation of curves.

Thesis 1 Geometric Hermite interpolation of curves:

1.1 Quantitative separation of geometric constraints and degrees of freedom of parametrization was given by deriving a recurrence formula that governs the change of Frenet coordinates of derivatives.

1.2 General formulation of arbitrary order geometric Hermite interpolation of curves was given and extended to reconstruction at multiple parameter values.
1.3 **Interpolant existence conditions** were derived that only depend on the basis function evaluation matrix of the representation and the geometric constraints. I presented upper bounds on the degree of integral polynomial GH interpolants.

The second thesis is the collection of my contributions that applied the results of Thesis 1 to construct geometric Hermite interpolants and utilize the degrees of freedom of parametrization.

**Thesis 2** *Algorithms of geometric Hermite interpolation of curves:*

2.1 Computation of approximate solutions to general GH problems were given using algebraic distance minimizing functionals (in the $L^1$ and $L^2$ norms) and functionals based on geometric quantities.

2.2 Incorporation of partial symmetric and asymmetric exact reconstruction constraints into approximate solutions were presented via transformation of the original GH reconstruction equations.

2.3 A two-level parametrization optimization framework was presented to optimize the parametrization of the curve in the sense of a general, real valued functional.

2.4 A geometric characterization of third order GH interpolation in Bernstein basis was given.

2.5 Blending based solutions to GH interpolation were presented. A Bézier curve blending based solution was given to match the polynomial degree of direct solutions obtained by matrix inversion.

The flexibility of the formulation of GH interpolation of curves is shown in Thesis 3, when the same formal structure is applied to GH interpolation of surfaces.

**Thesis 3** *Geometric Hermite interpolation of surfaces:*

3.1 I gave a geometric characterization of the generalization of second order GH interpolation to surfaces. Geometric conditions on input data were given on the existence of integral bi-quintic, bi-quartic, and rational bi-cubic Bézier patches.
3.2 I presented how geometric invariants of lines of curvature specify the higher order local geometry of a surface at a point, how parametrization alters the geometry of partial derivatives in comparison, and how geometric invariants of lines of curvature relate to geometric continuity conditions of surfaces.

3.3 Using the above results, I formalized general GH interpolation of surfaces such that two surfaces that reconstruct the same order n GH data tuple form a $G^n$ continuous join at the reconstruction point.

3.4 Interpolant existence conditions were derived that only depend on the basis function evaluation matrix of the representation and the geometric constraints. I presented upper bounds on the degree of integral polynomial GH interpolants.

GH interpolation of surfaces differs from the curve case in that the interpolants possess a considerable number of unconstrained control data that are not subject to reconstruction. Thesis 4 presented methods that supply values to these unconstrained control data as well, which could serve as initial values for parametrization optimization or be modified by the user.

**Thesis 4 Algorithms of geometric Hermite interpolation of surfaces:**

4.1 The facilities presented for computation of approximate solutions to GH interpolation and parametrization optimization of curves were extended to surfaces.

4.2 Direct and blending based algorithms were presented to create GH interpolant surface patches.

**Publications**


Conference talks


**Conference posters**


**References**


